

Enlacements asymptotiques revisités.

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Abstract

We give an alternative proof of a theorem of Gambaudo-Ghys [4] and Fathi [5] on the interpretation of the Calabi homomorphism for the standard symplectic disc as an average rotation number. This proof uses only basic complex analysis.

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1 The theorem of Gambaudo-Ghys and Fathi.

Let $\mathcal{G} = Ham_c(\mathbb{D}, \omega)$ be the group of compactly supported Hamiltonian diffeomorphisms of the standard disc $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ endowed with the standard symplectic form $\omega = \frac{i}{2} dz \wedge d\bar{z}$. The Calabi homomorphism [2] from \mathcal{G} to \mathbb{R} is defined as

$$Cal(\phi) = \int_0^1 dt \int_{\mathbb{D}} H_t \omega,$$

where H_t is the normalized Hamiltonian (zero near the boundary) of a Hamiltonian isotopy $\{\phi_t\}_{t \in [0,1]}$ with $\phi_1 = \phi$. In other words this isotopy is generated by a time-dependent vector field X_t , that satisfies the relation

$$\iota_{X_t} \omega = -dH_t.$$

The mean rotation number is defined in terms convenient for the proof as follows. Consider the differential form

$$\alpha = \frac{1}{2\pi} \frac{d(z_1 - z_2)}{z_1 - z_2}$$

(used by Arnol'd [1] in his study of the cohomology of the pure braid groups) on the configuration space $X_2 = X_2(\mathbb{D}) = \{(z_1, z_2) | z_j \in \mathbb{D}, z_1 \neq z_2\} = \mathbb{D} \times \mathbb{D} \setminus \Delta$, where $\Delta \subset \mathbb{D} \times \mathbb{D}$ is the diagonal. Denote by

$$\theta = Im(\alpha)$$

its imaginary part. Note that the two forms α and θ are closed. For each pair of points $(z_1, z_2) \in \mathbb{D} \times \mathbb{D}$ such that $z_1 \neq z_2$, that is for each point $x = (z_1, z_2) \in X_2$, consider the curve $\{\phi_t \cdot x\}$ in X_2 defined by

$$\phi_t \cdot x = (\phi_t(z_1), \phi_t(z_2))$$

for each $t \in [0, 1]$. The average rotation number is

$$\Phi(\phi) = \int_{X_2} dm^2(x) \int_{\{\phi_t \cdot x\}} \theta,$$

where $dm^2(x) = dm(z_1)dm(z_2)$ is the Lebesgue measure on $\mathbb{D} \times \mathbb{D}$ restricted to X_2 . By preservation of volume, it is clear that Φ is a homomorphism $\mathcal{G} \rightarrow \mathbb{R}$.

The theorem of Gambaudo-Ghys [4] and Fathi [5] is the following equality.

Theorem 1.

$$\Phi = -2Cal,$$

as homomorphisms $\mathcal{G} \rightarrow \mathbb{R}$.

Gambaudo and Ghys have presented several proofs of this result, and in [5] a different proof of Fathi is found. More proofs of this result are known today (cf. [3]). Here we present an alternative short proof, which is in fact a complex-variable version of the proof of Fathi.

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2 The alternative proof.

Put $\xi_t = dz(X_t)$, for the natural complex coordinate z on \mathbb{D} . Hence ξ_t is a smooth complex-valued function on \mathbb{D} , vanishing near the boundary. The computations

$$\iota_{X_t}(\frac{i}{2}dz \wedge d\bar{z}) = \frac{i}{2}\xi_t d\bar{z} - \frac{i}{2}\bar{\xi}_t dz$$

and

$$-dH_t = -\frac{\partial H_t}{\partial \bar{z}} d\bar{z} - \frac{\partial H_t}{\partial z} dz$$

give us

$$\xi_t = 2i \frac{\partial H_t}{\partial \bar{z}}. \tag{1}$$

Now

$$\Phi(\phi) = Im(\int_{X_2} dm^2(x) \int_{\{\phi_t \cdot x\}} \alpha),$$

and hence it is sufficient to compute

$$\begin{aligned} \int_{X_2} dm^2(x) \int_{\{\phi_t \cdot x\}} \alpha &= \frac{1}{2\pi} \int_{X_2} dm^2(x) \int_{\{\phi_t \cdot x\}} \frac{d(z_1 - z_2)}{z_1 - z_2} = \\ &= \frac{1}{2\pi} \int_{X_2} dm(z_1)dm(z_2) \int_0^1 dt \frac{\xi_t(\phi_t(z_1)) - \xi_t(\phi_t(z_2))}{\phi_t(z_1) - \phi_t(z_2)} = \end{aligned}$$

as the function is absolutely integrable (see Lemma 1), by Fubini,

$$= \frac{1}{2\pi} \int_0^1 dt \int_{X_2} dm(z_1)dm(z_2) \frac{\xi_t(\phi_t(z_1)) - \xi_t(\phi_t(z_2))}{\phi_t(z_1) - \phi_t(z_2)} = \frac{1}{2\pi} \int_0^1 dt \int_{X_2} dm(z_1)dm(z_2) \frac{\xi_t(z_1) - \xi_t(z_2)}{z_1 - z_2} =$$

as both terms of the sum are absolutely integrable (a consequence of the proof of Lemma 1 as well),

$$= 2 \cdot \frac{1}{2\pi} \int_0^1 dt \int_{\mathbb{D}} dm(w) \int_{\mathbb{D} \setminus \{w\}} \frac{\xi_t(z)}{z - w} dm(z) =$$

by Equation 1,

$$= -2i \int_0^1 dt \int_{\mathbb{D}} dm(w) \int_{\mathbb{D} \setminus \{w\}} \frac{1}{2\pi i} \frac{\partial H_t}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z - w} = -2i \int_0^1 dt \int_{\mathbb{D}} dm(w) H_t(w) = -2i \text{Cal}(\phi).$$

The penultimate equality is a consequence of the Cauchy formula for smooth functions [7, Theorem 1.2.1]. For any C^1 function $f : \mathbb{D} \rightarrow \mathbb{C}$, we have

$$f(w) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(z)}{z - w} dz + \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z - w}.$$

It remains to note that as H_t is zero near the boundary, the first term of the sum vanishes.

Now we show the absolute integrability that we use to change the order of integration.

Lemma 1.

$$\int_{X_2} \int_0^1 dm(z_1) dm(z_2) dt \frac{|\xi_t(\phi_t(z_1)) - \xi_t(\phi_t(z_2))|}{|\phi_t(z_1) - \phi_t(z_2)|} < \infty$$

By the Tonelli theorem, the following chain of inequalities suffices:

$$\begin{aligned} & \int_0^1 dt \int_{X_2} dm(z_1) dm(z_2) \frac{|\xi_t(\phi_t(z_1)) - \xi_t(\phi_t(z_2))|}{|\phi_t(z_1) - \phi_t(z_2)|} = \\ &= \int_0^1 dt \int_{X_2} dm(z_1) dm(z_2) \frac{|\xi_t(z_1) - \xi_t(z_2)|}{|z_1 - z_2|} \leq 2 \int_0^1 dt \int_{z \in \mathbb{D}} dm(z) |\xi_t(z)| \int_{w \in \mathbb{D} \setminus \{z\}} \frac{1}{|z - w|} dm(w) \leq \\ & \leq 8\pi \int_0^1 dt \int_{\mathbb{D}} |\xi_t| dm < \infty, \end{aligned}$$

because

$$\int_{w \in \mathbb{D} \setminus \{z\}} \frac{1}{|z - w|} dm(w) \leq 4\pi,$$

as one verifies by direct calculation.

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